AN APPROXIMATE SOLUTION FOR A STATIONARY HEAT-CONDUCTION PROBLEM IN A HALF-SPACE WITH MIXED BOUNDARY CONDITIONS

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A stationary axisymmetric problem with conditions of convective heat transfer is considered. A special method is employed to find an approximate solution in analytic form and its accuracy is evaluated.

We will examine the half-space $z > 0$ at whose boundaries (surrounded by a circle of unit radius) a constant temperature is maintained, with conditions of convective heat transfer maintained over the remainder of the boundary plane. In the case of a homogeneous and isotropic medium, determination of the temperature distribution in the half-space can be reduced, as is well known, to determination of the harmonic function $\theta(r, z)$ for boundary conditions of the form

$$\theta = 1 \text{ when } z = 0,\ r < 1, \tag{1}$$

and

$$\theta - k\frac{\partial\theta}{\partial z} = 0 \text{ when } z = 0,\ r > 1, \tag{2}$$

where $k = \text{const} > 0$. (Here and beyond, we will use dimensionless quantities).

We know of a number of references [1–3] in which this kind of problem is reduced to the solution of integral or integrodifferential equations. However, the results derived in these references permit only numerical solution, often significantly restricting the possibility of its interpretation. In this connection, we consider an approximate method of solving the stated problem, and one which makes it possible to derive an analytical expression for the solution.

We will present the unknown functions $\theta(r, z)$ in the form of the sum of the two functions

$$\theta = \theta' + \theta'',$$

satisfying the following boundary conditions when $z = 0$:

$$\theta' = \begin{cases} f(r), & r < 1, \quad (3)\\ 0, & r > 1, \quad (4)\end{cases}$$

$$\theta'' = 1 - f(r),\quad r < 1, \tag{5}$$

$$\frac{\partial\theta''}{\partial z} = 0,\quad r > 1. \tag{6}$$

Condition (1) is identically satisfied in this case for all $f(r)$, while in order to satisfy condition (2) it is necessary to determine $f(r)$ so as to satisfy the following equation when $r \geq 1$:

$$\theta''(r, 0) = k\frac{\partial\theta'(r, 0)}{\partial z}. \tag{7}$$

The problem of finding the harmonic functions θ' and θ'' involves no fundamental difficulties. The solution of the first of the indicated problems in this case can be obtained directly by means of the Green's function for a half-space and has the form

$$\theta'(r, z) = \frac{z}{2\pi}\int_0^{2\pi}\int_0^1 \frac{\rho f(\rho)\, d\rho\, d\psi}{(z^2 + r^2 + \rho^2 - 2r\rho\cos\psi)^{3/2}}, \tag{8}$$

while the solution of the second problem is presented in the form

$$\theta''(r, z) = \int_0^\infty A(\lambda)\exp(-\lambda z) J_0(\lambda r)\, d\lambda, \tag{9}$$

where $A(\lambda)$ is found from the solution of the paired integral equations:

$$\int_0^\infty A(\lambda) J_0(\lambda r)\, d\lambda = 1 - f(r),\quad r < 1, \tag{10}$$

$$\int_0^\infty \lambda A(\lambda) J_0(\lambda r)\, d\lambda = 0,\quad r > 1. \tag{11}$$

To find the function $A(\lambda)$ we can use, for example, the familiar method proposed in [4]. By presenting $A(\lambda)$ in the form

$$A(\lambda) = \int_0^1 \varphi(t)\cos\lambda t\, dt \tag{12}$$

we can satisfy Eq. (11) identically, while Eq. (10) is brought to the form of the Schloemilch integral equation

$$\int_0^r \frac{\varphi(t)\, dt}{\sqrt{r^2 - t^2}} = 1 - f(r), \tag{13}$$

whose solution is known [5] and is given by the expression

$$\varphi(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{r\,[1 - f(r)]}{\sqrt{t^2 - r^2}}\, dr. \tag{14}$$

The function θ'' is thus defined by expressions (9), (12), and (14), which, as does (8), include the unknown function $f(r)$. Determining $f(r)$ directly by means of relation (7) is difficult, since substitution into the latter of the derived expressions for θ' and θ'' transforms it into the Fredholm integral equation of the I-st kind with a nonsymmetric kernel.

To find the approximate solution of the initial problem, we assume a certain expression of $f(r)$ so as to satisfy the following conditions:

1) the unknown function θ must be continuous everywhere at the boundary and together with the derivative $\partial\theta/\partial z$ it must diminish monotonically with increasing r (when $r > 1$);

2) the total heat flow from the boundary plane must equal zero, i.e.,

$$\int_0^\infty \frac{\partial\theta(r,\ 0)}{\partial z} r dr = 0;$$

3) condition (7) must be satisfied at least when $r = 1$.

The formulated requirement can be satisfied if we assume $f(r)$ in the form

$$f(r) = \begin{cases} B, & r < c < 1, \\ 0, & c < r < 1, \end{cases} \tag{15}$$

where B and c are parameters.

The first of the above-cited conditions, as is not difficult to see, is satisfied for any values of the parameters.

Substitution of (15) into (8) leads to the following expression (see, for example, [7]):

$$\frac{\theta'(r,\ z)}{B} = 1 - \frac{z}{\pi\sqrt{z^2 + (r+c)^2}} \times \left[\frac{a-c}{a+r}\,\Pi\left(\frac{\pi}{2},\ n_1,\ \nu\right) + \frac{a+c}{a-r}\,\Pi\left(\frac{\pi}{2},\ n_2,\ \nu\right)\right], \tag{16}$$

where $a = (r^2 + z^2)^{1/2}$, $n_1 = -2r/a + r)$, $n_2 = 2r/a - r)$, and $\nu^2 = 4cr/[(r + c)^2 + z^2]$. Analogously, by means of relations (14), (12), and (9), we obtain [6]

$$\varphi(t) = \begin{cases} \frac{2}{\pi}(1-B), & t < c, \\ \frac{2}{\pi}\left[1 - B\left(1 - \frac{t}{\sqrt{t^2 - c^2}}\right)\right], & t > c, \end{cases}$$

$$A(\lambda) = \frac{2}{\pi}\left[(1-B)\frac{\sin\lambda}{\lambda} + B\int_c^1 \frac{t\cos\lambda t\, dt}{\sqrt{t^2 - c^2}}\right],$$

$$\theta''(r,\ z) = \frac{2}{\pi}\left[(1-B)\int_0^\infty \frac{\exp(-\lambda z)}{\lambda} J_0(\lambda r)\sin\lambda\, d\lambda + B\int_c^1 \frac{t dt}{\sqrt{t^2-c^2}}\int_0^\infty J_0(\lambda r)\exp(-\lambda z)\cos\lambda t\, d\lambda\right] =$$

$$= \frac{2}{\pi}\left[(1-B)\,\mathrm{arctg}\frac{\sqrt{2} + \left[\sqrt{(z^2+r^2-1)^2+4z^2} - (z^2+r^2-1)\right]^{1/2}}{\sqrt{2}z + \left[\sqrt{(z^2+r^2-1)^2+4z^2} + (z^2+r^2-1)\right]^{1/2}} - \frac{B}{\sqrt{2}}\int_c^1 \frac{t\left[\sqrt{(z^2+r^2-t^2)^2+4z^2t^2} + z^2 + r^2 - t^2\right]^{1/2}}{\sqrt{(t^2-c^2)\left[(r^2+z^2-t^2)^2+4z^2t^2\right]}}dt\right]. \tag{17}$$

The relationship between the parameters B and c can be established by proceding from the condition that the total heat flow be equal to zero, which is equivalent to the case in which the coefficient for $1/z$ in the expansion of $\theta(0, z)$ in powers of $1/z$ is equal to 0 in the vicinity of an infinitely remote point. Assuming in (16) and (17) that $r = 0$ and carrying out the integration in (17), we derive the following expression:

$$\theta(0,\ z) = B\left(1 - \frac{1}{\sqrt{1 + \frac{c^2}{z^2}}}\right) + \frac{2}{\pi}\left[(1-B)\,\mathrm{arctg}\frac{1}{z} + \frac{Bz}{\sqrt{c^2+z^2}}\,\mathrm{arctg}\frac{\sqrt{1-c^2}}{\sqrt{c^2+z^2}}\right] \tag{18}$$

Expanding (18) in a power series and equating the coefficient for $1/z$ to zero, we obtain the relationship

$$B = \frac{1}{1 - \sqrt{1-c^2}}. \tag{19}$$

Substituting (19) into (16) and (17), by means of appropriate transformations we obtain an expression for the temperature and density of the heat flow at the boundary when $r > 1$:

$$\theta(r,\ 0)|_{r>1} = \frac{2}{\pi}\,\frac{1}{1-\sqrt{1-c^2}}\left(\frac{\pi}{4} + \frac{1}{2}\arcsin\frac{2 - r^2 - c^2}{r^2 - c^2} - \sqrt{1-c^2}\,\mathrm{arctg}\frac{1}{\sqrt{r^2-1}}\right), \tag{20}$$

$$\left.\frac{\partial\theta(r,\ 0)}{\partial z}\right|_{r>1} = = \frac{2}{\pi}\,\frac{1}{1-\sqrt{1-c^2}}\left[\frac{rE\left(\frac{c}{r}\right)}{r^2-c^2} - \frac{K\left(\frac{c}{r}\right)}{r}\right]. \tag{21}$$

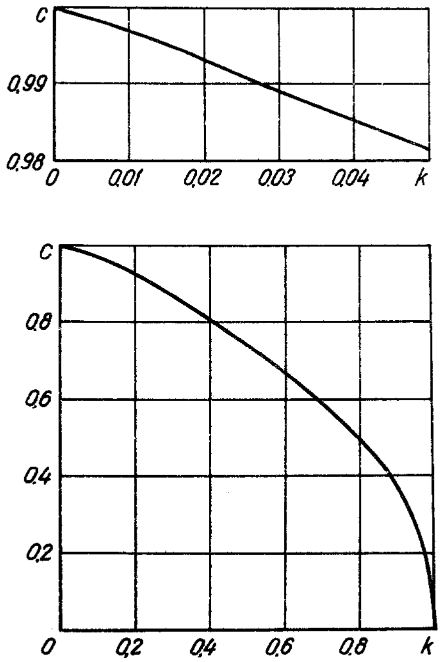
Proceeding from (21), for the total heat flow we have

$$Q = 2\pi\int_1^\infty \frac{\partial\theta}{\partial z} r dr = 4\,\frac{K(c) - E(c)}{1 - \sqrt{1-c^2}}. \tag{22}$$

Expression (22) can also be used to find heat capacity.

In all of the results obtained above we have the parameter c for whose determination we use the third of the above-formulated conditions, i.e., we determine this parameter from the relationship

$$\theta''(1,\ 0) = k\frac{\partial\theta'(1,\ 0)}{\partial z}. \tag{23}$$



Graph for determination of parameter c.

Magnitude of the Deviation Between the Approximate Boundary Conditions when $r > 1$ (α_{max})

r	k				
	0.97	0.60	0.30	0.10	0.05
1 – 1.2	0.28	0.23	0.19	0.08	0.02
1.2 – 1.5	0.24	0.18	0.10	0.01	0.01
1.5 – 2.0	0.15	0.10	0.04	<0.01	<0.01
2.0 – 5.0	0.05	0.03	0.01	<0.01	<0.01
>5.0			<0.01		

Having substituted (20) and (21) into (23), we obtain the relationship between the known quantity k and the parameter c

$$k = \frac{\pi}{2} \frac{(1-c^2)(1-\sqrt{1-c^2})}{E(c)-(1-c^2)K(c)}. \quad (24)$$

It follows from (24) that the change in the parameter c in the specified interval [0, 1] corresponds to a change in k in the interval [1, 0], for which the derived solution of the problem is consequently valid.

The magnitude of the parameter c for the specified $k \in [0, 1]$ can be determined in this case by proceeding from the relationship calculated with (24) and shown in the figure.

The accuracy of the derived approximate solution is determined from the deviation of the boundary condition from the specified condition (2) when $r > 1$:

$$\alpha(r) = \theta(r, 0) - k\frac{\partial\theta(r, 0)}{\partial z}, \quad 1 < r < \infty. \quad (25)$$

(In particular, when $k = 0$ ($c = 1$), we have $\alpha(r) = 0$, i.e., the derived solution is exact in this case). The calculation of $\alpha(r)$ showed that the greatest deviation is encountered in values of r close to unity, where this function exhibits an extremum. The table below gives the largest absolute values of α_{max} for various intervals of variation in r for various values of k.

The upper limit of the error in the approximate solution derived by us can be estimated to an accuracy of 0.01 at any point of the half-space $z > 0$ by means of the auxiliary function $\Delta(r, z)$, if the latter is harmonic and if it assumes the values of α_{max} when $z = 0$:

$$\Delta(r, 0) = \begin{cases} 0; & r < 1; \\ 0.28; & 1 < r < 1.5; \\ 0.15; & 1.5 < r < 2.0; \\ 0.05; & 2.0 < r < 5.0; \\ 0; & 5 < r < \infty. \end{cases} \quad (26)$$

The determination of the function $\Delta(r, z)$, reducing to the solution of the Dirichlet problem for a half-space with boundary condition (26), involves no difficulties; when $r = 0$, at the axes of the system, the solution has the form

$$\Delta(0, z) = z\left(\frac{0.28}{\sqrt{1+z^2}} - \frac{0.13}{\sqrt{2{,}25+z^2}} - \frac{0.10}{\sqrt{4+z^2}} - \frac{0.05}{\sqrt{25+z^2}}\right). \quad (27)$$

The calculation carried out in accordance with formula (27) demonstrated that the error in the solution of the original problem for the points on the axes of the system does not exceed 0.07, i.e., it amounts to no more than several percent of the maximum value of the function at the boundary.

The derived approximate solution for the problem with boundary conditions (1) and (2) thus yields completely satisfactory accuracy.

In conclusion, it should be noted that the above-cited results can also be used for similar problems in potential theory and, in particular, in calculating the electric field of linearly polarizing electrodes.

NOTATION

θ is the temperature; Q is the heat flux; f and φ are function symbols; α is the difference in boundary conditions; Δ is the error in calculation; ρ and t are integration variables; λ is the parameter of variables division in the Laplace equation; k is a real constant; B and c are parameters; r and z are cylindical coordinates; J_0 and J_1 are Bessel functions of the first kind of zeroth and first order, respectively; K, E, and II are total elliptic integrals of the first, second, and third kind, respectively.

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